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EXISTENCE, UNIQUENESS AND STABILITY OF SOLUTIONS OF A CLASS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS

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INTRODUCTION

In this work we present a unified approach for treating the existence, uniqueness and asymptotic stability of classical solutions for a class of nonlinear partial differential equations governing the behavior of nonlinear continuous dynamical systems. From this class we treat the following initial boundary value problems for α a positive constant in varying detail:

$$u_{tt} - 2\alpha u_{xxt} - u_{xx} = f(u,u_x,u_t,u_{xt},u_{xx},x,t) \tag{A}$$

$$u_{tt} + 2\alpha u_t - u_{xx} = f(u,u_x,u_t,x,t) \tag{B}$$

$$u_t - u_{xx} = f(u,u_x,x,t) \tag{C}$$

$$u_{tt} - 2\alpha\nabla^2 u_t - \nabla^2 u = f(u,u_x,u_y,u_t,x,t) \tag{D}$$

$$\left.\begin{aligned} u_{tt} - 2\alpha u_{xxt} - u_{xx} &= f_1(u,v,u_x,v_x,u_t,v_t,u_{xx},v_{xx},u_{xt},v_{xt},x,t) \\ v_{tt} - 2\alpha v_{xxt} - v_{xx} &= f_2(u,v,u_x,v_x,u_t,v_t,u_{xx},v_{xx},u_{xt},v_{xt},x,t) \quad . \end{aligned}\right\} \tag{E}$$

Here there is no intention of indicating how the content of the nonlinearities is determined. That will be discussed in our detailed consideration of (A).

Some problems of this type have been treated before. Ficken and Fleishman [1] investigated the existence, uniqueness and stability of solutions of the initial value problem for

$$u_{xx} - u_{tt} - 2\alpha_1 u_t - \alpha_2 u = \epsilon u^3 + b \ .$$

Greenberg, MacCamy and Mizel [2] have treated the initial boundary value problem for $u_{tt} - u_{xxt} = \sigma'(u_x)u_{xx}$ (which is a special case of (A)) using some results from the theory of parabolic equations. Rabinowitz [3,4] has proven the existence of periodic solutions for $u_{tt} + 2\alpha u_t - u_{xx} = \epsilon f$ where ϵ is a small parameter and f is periodic in time. In [3] he treats $f = f(u,u_x,u_t,x,t)$ and in [4] he treats the fully nonlinear case $f = f(u,u_x,u_t,u_{xx},u_{xt},u_{tt},x,t)$. To do this he uses methods from the theory of elliptic boundary value problems. Avner Friedman treats equations of the type (C) in his book on parabolic equations [5]. Notice that Burger's equation is a special case of (C).

This paper is divided into two parts. In part one we discuss the general ideas involved, including the general type of problem to which the techniques of this paper can be applied, the fixed point theorem used for proving existence and uniqueness, and a Liapunov functional approach to stability which is an extension of Liapunov's direct method for ordinary differential equations. In part two we treat the specific equations listed above in varying detail, using the ideas developed in part one.

Equation (A) will be treated in the greatest detail, making concrete all the basic ideas of this paper. Equations (B) and (C) will be discussed in a similar manner; however, the lemmas and proofs of the theorems will not be included. Because of space limitations, we give only a brief discussion of equation (D); it was selected because it has two spatial dimensions. Equation (E) will not be considered at all, it can be treated in the same way as (A) and is mentioned here only to make the reader aware of the applicability of the ideas in this paper to systems. We feel that an understanding of part two will give the reader the necessary tools for handling similar problems.

Much of this material was originally discussed in [6] in a Sobolev space context; this paper is an improvement over those results. The authors thank Charles De Prima of Cal Tech for the suggestion that led to the present treatment.

PART I

GENERAL THEORY

1.1 INTRODUCTION

In this part we discuss the basic ideas for treating existence, uniqueness and stability for problems of the form:

$$\mathcal{L}u \equiv u_{tt} + 2\alpha\mathcal{L}_1 u_t + \mathcal{L}_2 u = f \tag{1}$$

$$\alpha > 0 , \quad x \in \mathcal{D} \text{ and } t \in [0,T]$$

with homogeneous boundary conditions

$$Bu(x,t) = 0 \quad \text{for} \quad x \in \partial\mathcal{D} \tag{2}$$

and initial conditions

$$u(x,0) = a_1(x), \quad u_t(x,0) = a_2(x) . \tag{3}$$

Here $\mathcal{L}_1$ and $\mathcal{L}_2$ are linear self adjoint "spatial" operators with certain other properties depending on the context, f is a nonlinear function of x, t, u and some derivatives of u, and $\mathcal{D}$ is a spatial domain.

In the treatment of equations (A),(B) and (C) in part II, $\mathcal{D}$ is the interval $[0,1]$ and for equation (D) it is $[0,1] \times [0,1]$. Equation (C) does not fall into the form of (1)-(3), however, it will be clear how to extend the ideas of this paper to the initial boundary value problem:

$$\left.\begin{array}{l} u_t + \mathcal{L}u = f, \quad x \in \mathcal{D} \quad \text{and} \quad t \in [0,T] \\ \text{with homogeneous boundary conditions} \\ \qquad Bu(x,t) = 0 \quad \text{for} \quad x \in \partial\mathcal{D} \\ \text{and initial condition} \\ \qquad u(x,0) = a(x) , \\ \text{of which (C) is a special case.} \end{array}\right\} \tag{4}$$

1.2 FORMULATION OF EXISTENCE AND UNIQUENESS FOR (1) - (3).

The key idea in our method for proving existence and uniqueness is the construction of a solution to the linear nonhomogeneous problem

$$\mathcal{L}u = F(x,t) \tag{5}$$

associated with (1) by the use of an eigenfunction expansion. In order to do this we assume there exists a complete set of orthonormal eigenfunctions $\{\varphi_n(x)\}$ which are eigenfunctions for both $\mathcal{L}_1$ and $\mathcal{L}_2$, i.e., $\mathcal{L}_1\varphi_n = \lambda_n\varphi_n$, $\mathcal{L}_2\varphi_n = \mu_n\varphi_n$, and $B\varphi_n(x) = 0$ for $x \in \partial\mathcal{D}$. Caughey and O'Kelly [7] have derived necessary and sufficient conditions for this to be true; we state them here without proof:

a. the operators $\mathcal{L}_1$ and $\mathcal{L}_2$ commute, i.e., $\mathcal{L}_1\mathcal{L}_2 = \mathcal{L}_2\mathcal{L}_1$,

b. if the operators are of different order the boundary conditions on the higher order operator must be derivable from a compatible set of boundary conditions on the lower order operator.

The unique solution of (1)-(3) can be viewed as a fixed point of the mapping $A: u \to v$ defined by

$$\mathcal{L}v = f(u,u_t,\ldots,x,t) \tag{6}$$

or in terms of A

$$v = Au \tag{7}$$

where u and v are required to satisfy the initial and boundary conditions (2) and (3). This, of course, is not the only mapping which can be defined from (1); however, this form is particularly useful because of the "damping" term, $\mathcal{L}_1 v_t$, and because of the properties of the eigenvalues of $\mathcal{L}_1$ and $\mathcal{L}_2$.

Since $\mathcal{L}_1$ and $\mathcal{L}_2$ are assumed to have the same complete set of eigenfunctions, an eigenfunction expansion can be used to find an explicit representation for the mapping. This is the same as saying that the linear nonhomogeneous equation (6) with $F(x,t) = f(u(x,t),u_t(x,t),\ldots,x,t)$ can be solved by an eigenfunction expansion

$$v(x,t) = \sum_1^\infty v_n(t)\varphi_n(x) \ . \tag{8}$$

The differential equation (6) requires that the $v_n(t)$ must satisfy the nonhomogeneous ordinary differential equation

$$v_{ntt} + 2\alpha\lambda_n v_{nt} + \mu_n v_n = F_n(t) \tag{9}$$

with initial conditions

$$v_n(0) = a_{1n} \ , \quad v_{nt}(0) = a_{2n} \tag{10}$$

where

$$F_n(t) = \int_{\mathcal{D}} F(x,t)\varphi_n(x)dx = \int_{\mathcal{D}} f(u(x,t),u_t(x,t),\ldots,x,t)\varphi_n(x)dx \tag{11}$$

and

$$a_{in} = \int_{\mathcal{D}} a_i(x)\varphi_n(x)dx \ , \quad i = 1,2 \ . \tag{12}$$

The solution of (9)-(10) can be written

$$v_n(t) = a_{1n}v_{1n}(t) + a_{2n}v_{2n}(t) + \int_0^t F_n(\nu)v_{2n}(t-\nu)d\nu \tag{13}$$

where $v_{1n}(t)$ and $v_{2n}(t)$ solve the homogeneous equation (9) with initial conditions

$$\begin{aligned} v_{1n}(0) &= 1 \quad v_{2n}(0) = 0 \\ v_{1nt}(0) &= 0 \quad v_{2nt}(0) = 1 \ . \end{aligned} \tag{14}$$

The mapping (7) can now be written

$$v = Au = \sum_1^\infty a_{1n}v_{1n}(t)\varphi_n(x) + \sum_1^\infty a_{2n}v_{2n}(t)\varphi_n(x)$$
$$+ \sum_1^\infty \left(\int_0^t F_n(\nu)v_{2n}(t-\nu)d\nu\right)\varphi_n(x) \; . \tag{15}$$

This is the form of the mapping we use in our discussion of existence and uniqueness.

The existence and uniqueness of a classical solution to (1)-(3) is thus reduced to showing that A, explicitly given in (15), is a contraction mapping on a suitable complete metric space. The distance function we use is a curious sort of norm; for example, in the case of equation (A) we use the distance function $d(u_1,u_2) = \|u_1-u_2\|$ where

$$\|u\| = |u|_m + |u_x|_m + |u_t|_m + |u_{xx}|_m + |u_{xt}|_m$$
$$+ |u_{tt}|_m + |u_{xxx}|_m + |u_{xxt}|_m + |u_{xxxx}|_{Lm} \tag{16}$$
$$+ |u_{xxxt}|_{Lm} + |u_{xtt}|_{Lm}$$

and

$$|g(x,t)|_m = \max_\Omega |g(x,t)|, \quad \Omega = [0,1] \times [0,T] \tag{17}$$
$$|g(x,t)|_{Lm} = \max_{t\in[0,T]} \left(\int_0^1 g^2(x,t)dx\right)^{1/2} .$$

The form of the fixed point theorem we use is taken from Korevaar [8], page 213:

<u>Definition</u>: Let N be a metric space. The transformation A of N into itself is called a contraction if there exists a positive constant $r < 1$ such that

$$d(Au_1,Au_2) \le r\,d(u_1,u_2) \quad \text{for all } u_1,u_2 \text{ in } N .$$

Contraction Mapping Theorem: Let A be a contraction operator on a nonempty complete metric space N. Then A has exactly one fixed-point u, and if u_0 is any point in N

$$u = \lim_{k\to\infty} A^k u_0 .$$

As $k \to \infty$ the distance between u and $A^k u_0$ tends to zero at least as fast as r^k.

The proof of this is in Korevaar.

In each case in part II we proceed by defining a suitable complete metric space and then finding conditions such that A maps the space into itself and is a contraction. Each case contains three results related to existence and uniqueness:

(a) An existence, uniqueness theorem on a finite interval $t \in [0,T]$. Here there are essentially no restrictions on f, a_0 and a_1 except for smoothness requirements, and the contraction mapping is obtained by making T small. The problem of the extension of solutions to a maximum time interval is not considered in this paper.

(b) An existence, uniqueness theorem on the semi-infinite interval $t \in [0,\infty)$. Here along with smoothness conditions we require that a_0, a_1 and f be small and that $f|_{u=0} = 0$ (i.e. $u = 0$ is an equilibrium solution). This theorem also gives the asymptotic stability of the zero solution.

(c) A bound on the solution under the conditions used in (b). This is obtained by using a form of the Gronwall lemma.

Since nonlinear problems are in general much more difficult than linear problems, an interesting question is "When does the solution of the nonlinear problem behave like the solution of the linearized version?"

For ordinary differential equations there are theorems attributed in various places to Liapunov, Poincare and Perron, which say, in essence, that if solutions of the linearized equation are asymptotically stable and the nonlinearity is small then solutions of the nonlinear equation are asymptotically stable. Results (b) and (c) together comprise a theorem of this type for the equations considered in this paper; a Liapunov-Poincare type theorem.

1.3 FORMULATION OF LIAPUNOV'S DIRECT METHOD FOR STABILITY.

Stability analysis by Liapunov's Direct Method has been applied extensively to ordinary differential equations and so it is natural to look for extensions of this method to partial differential equations. Several recent papers [9,10,11] treat stability for certain partial differential equations by such an extension. Greenberg, MacCamy and Mizel [2] and Rabinowitz [3] treat stability by a method which is essentially the same as the Direct Method.

In this section we state a theorem on asymptotic stability which applies to (1)-(2) if a "Liapunov Functional" can be constructed. It is assumed here that (1) admits a zero (equilibrium) solution.

We introduce a state vector $U(x,t)$ which consists of u defined by (1)-(3) (or (4)), u_t and various spatial derivatives of u and u_t, sometimes writing $U(x,t,U_0)$ where $U_0(x)$ denotes the initial state of the system, i.e., $U(x,0,U_0) = U_0$. The asymptotic stability will be discussed in terms of a time dependent norm,

$$\rho(t) = \left(\int_{\mathcal{D}} (U_1^2 + U_2^2 + \cdots + U_n^2)dx\right)^{1/2}, \tag{18}$$

where the $U_i(x,t)$ are the scalar components of U. For example, in Case A, $\rho(t) = \left(\int_0^1 (u^2+u_x^2+u_t^2+u_{xx}^2)dx\right)^{1/2}$.

Before proceeding to the theorem, which is almost identical to a theorem proved by Kalman and Bertram [12] for ordinary differential equations, we need the following stability definitions in terms of the norm, ρ:

<u>Definition 1</u>: The zero solution of (1)-(3) (or (4)) is said to be stable if for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that $\rho(0) \leq \delta$ implies $\rho(t) \leq \epsilon$ for all $t \geq 0$.

Definition 2: The zero solution of (1) - (3) (or (4)) is said to be asymptotically stable in the large if

1) the zero solution is stable, and
2) all solutions which are bounded initially ($\rho(0)$ is bounded) remain bounded for all time ($\rho(t)$ is bounded for all $t \geq 0$) and approach zero as $t \to \infty$ ($\lim_{t\to\infty} \rho(t) = 0$).

Let $\mathscr{V}$ be a spatial integral operator (functional) which maps the vector function $U(x,t)$ into a scalar function $V(t)$, i.e., $\mathscr{V}[U(x,t)] = V(t)$.

Liapunov Stability Theorem (Liapunov's Direct Method for Asymptotic Stability): Suppose there exists a functional $\mathscr{V}[U(x,t)] = V(t)$ differentiable in t along every solution curve U such that $\mathscr{V}[0] = 0$ and

a) $\mathscr{V}[U] = V(t)$ is positive definite, that is, there exists a continuous nondecreasing scalar function β_1 such that $\beta_1(0) = 0$ and for all t and all $U \not\equiv 0$, $0 < \beta_1(\rho(t)) \leq V(t)$;

b) there exists a continuous scalar function γ such that $\gamma(0) = 0$ and the derivative $\dot{V}$ of V along the motion satisfies, for all $t > 0$ and $U \not\equiv 0$, $\dot{V}(t) \leq -\gamma(\rho(t)) < 0$;

c) there exists a continuous, nondecreasing scalar function β_2 such that $\beta_2(0) = 0$ and, for all t, $V(t) \leq \beta_2(\rho(t))$;

d) $\beta_1(\rho) \to \infty$ as $\rho \to \infty$.

Then the zero solution of (1)-(3) (or (4)) is asymptotically stable in the large.

The proof is contained in [6] and a similar proof is contained in [12].

Definition 3: A functional $\mathscr{V}[U]$, which maps the vector function $U(x,t)$ into a scalar function of t and which satisfies the conditions of the above theorem is called a Liapunov Functional.

Liapunov's Direct Method for stability has the advantage that it does not require any knowledge of the solution (except that it satisfy a certain differential equation); however, it suffers in that there is no general way to find a Liapunov Functional. An exception to this is the case of (1) where $f \equiv 0$, i.e. the linear homogeneous case. A Lipaunov Functional can be constructed by multiplying $u_{tt} + 2\alpha\mathcal{L}_1 u_t + \mathcal{L}_2 u = 0$ by $u_t + \alpha\mathcal{L}_1 u$, integrating over the domain $\mathcal{D}$ and then making appropriate restrictions on $\mathcal{L}_1$ and $\mathcal{L}_2$. We shall demonstrate the procedure here since it can be carried out for the general operators $\mathcal{L}_1$ and $\mathcal{L}_2$ and because the functional so constructed will be useful for the cases considered in part II. In the following derivation we assume $\mathcal{L}_1$ and $\mathcal{L}_2$ are self-adjoint.

If the inner product of $f(x)$ and $g(x)$ is defined by

$$(f,g) = \int_{\mathcal{D}} fg\, dx$$

then

$$\begin{aligned} 0 = (u_t+\alpha\mathcal{L}_1 u, 0) &= (u_t+\alpha\mathcal{L}_1 u, u_{tt}+2\alpha\mathcal{L}_1 u_t+\mathcal{L}_2 u) \equiv \tfrac{1}{2}(u_t,u_t)_t \\ &+ (u_t, 2\alpha\mathcal{L}_1 u_t) + (u_t,\mathcal{L}_2 u) + (\alpha\mathcal{L}_1 u, u_{tt}) + (2\mathcal{L}_1 u, 2\alpha\mathcal{L}_1 u_t) + (\alpha\mathcal{L}_1 u, \mathcal{L}_2 u). \end{aligned} \tag{19}$$

The relations

$$\begin{aligned} (u_t,\mathcal{L}_2 u) &= \tfrac{1}{2}(u,\mathcal{L}_2 u)_t \\ (\alpha\mathcal{L}_1 u, u_{tt}) &= (\alpha\mathcal{L}_1 u, u_t)_t - (\alpha\mathcal{L}_1 u_t, u_t) \\ (\mathcal{L}_1 u, \mathcal{L}_1 u_t) &= \tfrac{1}{2}(\mathcal{L}_1 u, \mathcal{L}_1 u)_t \end{aligned} \tag{20}$$

imply

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[(u_t,u_t) + (u,\mathcal{L}_2 u) + 2\alpha(\mathcal{L}_1 u, u_t) + 2\alpha^2(\mathcal{L}_1 u, \mathcal{L}_1 u)\right] \\ &= -\frac{1}{2}\left[(u_t, 2\alpha\mathcal{L}_1 u_t) + (2\alpha\mathcal{L}_1 u, \mathcal{L}_2 u)\right]. \end{aligned} \tag{21}$$

If we define

$$V(t) = \frac{1}{2}\,[(u,\mathcal{L}_2 u) + (u_t+\alpha\mathcal{L}_1 u, u_t+\alpha\mathcal{L}_1 u) + \alpha^2(\mathcal{L}_1 u,\mathcal{L}_1 u)] \tag{22}$$

then (21) can be written

$$\dot{V}(t) = -\alpha[(u_t,\mathcal{L}_1 u_t) + (\mathcal{L}_1 u,\mathcal{L}_2 u)]\ . \tag{23}$$

By making appropriate restrictions on $\mathcal{L}_1$ and $\mathcal{L}_2$, e.g., positive definiteness, it can be shown that V in (22) is a Liapunov Functional for (1)-(2) with $f \equiv 0$.

If we consider (1) with $f \not\equiv 0$ the same procedure yields

$$V_1(t) = \frac{1}{2}\,[(u,\mathcal{L}_2 u)+(u_t+\alpha\mathcal{L}_1 u,\ u_t+\alpha\mathcal{L}_1 u) + \alpha^2(\mathcal{L}_1 u,\mathcal{L}_2 u)] \tag{24}$$

and

$$\dot{V}_1(t) = -\alpha[(u_t,\mathcal{L}_1 u_t)+(\mathcal{L}_1 u,\mathcal{L}_2 u)] + (u_t+\alpha\mathcal{L}_1 u,f)\ . \tag{25}$$

V_1 will be part of the Liapunov Functionals constructed in part II.

In the cases to follow later we pick one or two rather specific nonlinearities f and prove the asymptotic stability in the large in terms of some norm, ρ, by constructing a Liapunov functional. It can be seen from these cases that the application of the Liapunov stability theorem is more general than the Liapunov-Poincare result described in section 1.2 in the sense that there is no restriction on the size of the initial data (recall asymptotic stability in the large) or the size of the nonlinearity, only a restriction on the form of the nonlinearity. For example, in the Liapunov-Poincare type result the sign of the nonlinearity plays no role; however, in Liapunov's direct method it may determine the difference between stability or instability.

PART II
APPLICATIONS

2.1 INTRODUCTION.

Each of the four cases presented here is discussed in the following way: a) Existence and uniqueness of classical solutions (i.e. solutions such that u and all the derivatives appearing in the differential equation are continuous) on a finite interval and an infinite interval, b) a Liapunov-Poincare type theorem and c) stability via Liapunov's direct method. Case A is worked out in considerable detail and is the only case where proofs are given. We feel that an understanding of the proofs in Case A will allow the reader to supply his own proofs for the other cases. Cases B and C will be discussed in a similar manner; however, the lemmas and proofs of the theorems will not be included. For Case D, we give only a brief discussion.

2.2 CASE A.

2.2a. Preliminaries.

The differential equation is

$$\mathcal{L}u \equiv u_{tt} - 2\alpha u_{xxt} - u_{xx} = f(u,u_x,u_t,u_{xx},u_{xt},x,t)$$
$$x \in [0,1], \quad t \geq 0, \quad \alpha > 0, \tag{A1}$$

with boundary and initial conditions

$$u(0,t) = u(1,t) = 0, \tag{A2}$$

$$u(x,0) = a_1(x), \quad u_t(x,0) = a_2(x). \tag{A3}$$

The eigenfunctions and eigenvalues associated with $\mathcal{L}_1 = \mathcal{L}_2 = -\frac{\partial^2}{\partial x^2}$ (see section 1.2) are

$$\varphi_n(x) = \sqrt{2}\sin n\pi x, \quad \lambda_n = \mu_n = n^2\pi^2, \tag{A4}$$

so the solution of (A1)-(A3) can be viewed as a fixed point of the mapping A defined in equation (15) with

$$F(x,t) = f(u(x,t),u_x(x,t),u_t(x,t),u_{xt}(x,t),u_{xx}(x,t),x,t) \tag{A5}$$

and

$$\begin{aligned} v_{1n}(t) &= e^{-\alpha n^2\pi^2 t}\left[\cosh Xt + \frac{\alpha n^2\pi^2}{X}\sinh Xt\right] \\ v_{2n}(t) &= e^{-\alpha n^2\pi^2 t}\,\frac{\sinh Xt}{X} \end{aligned} \tag{A6}$$

where

$$X = (\alpha^2 n^4\pi^4 - n^2\pi^2)^{1/2} . \tag{A7}$$

The following relationships are needed for the existence, uniqueness discussion:

$$\begin{aligned} \int_0^\infty v_{2n}^2(t)dt &= \frac{1}{4\alpha n^2\pi^4} \\ \int_0^\infty v_{2nt}^2(t)dt &= \frac{1}{4\alpha n^2\pi^2} . \end{aligned} \tag{A8}$$

For every $\alpha > 0$ there exist positive a and k such that

$$\begin{aligned} |v_{1n}(t)| &\le ke^{-at} \\ |v_{2n}(t)| &\le \frac{ke^{-at}}{n^2\pi^2} \\ \left|\frac{d^i v_{jn}(t)}{dt^i}\right| &\le k(n\pi)^{2(i-1)}e^{-at}, \quad j = 1,2;\ i \ge 1 . \end{aligned} \tag{A9}$$

Also

$$\begin{aligned} \int_0^\infty (e^{at}v_{2n}(t))^2 dt &\le \frac{k^2}{n^4\pi^4} \\ \int_0^\infty (e^{at}v_{2nt}(t))^2 dt &\le \frac{k^2}{n^2\pi^2} . \end{aligned} \tag{A10}$$

The inequalities (A10) do not follow from (A9), but can be obtained by examining the integrals directly.

We want to prove existence and uniqueness of classical solutions to (A1)-(A3), that is, we seek solutions $u(x,t)$ such that u_{tt}, u_{xxt}, u_{xx} and u_{xt} are continuous. Before defining the appropriate norm and function spaces we offer the following partial motivation for our choice of norm (other norms were tried but failed to produce the desired contraction mappings).

Because of the nature of the Fourier series representation of solutions to $\mathcal{L}u = F(x,t)$, the square integrability of F_{xx} and F_t on $\Omega = [0,1] \times [0,T]$ is needed in order for u_{tt}, u_{xxt} and u_{xx} to be continuous on Ω (for details, see Lemma A2, following). The square integrability of F_{xx} and F_t also yields the continuity of u_{xxx} on Ω, the square integrability of u_{xxxx}, u_{xxxt} and u_{xtt} on $[0,1]$ for every t in $[0,T]$ and the continuity of these integrals in t. This indicates that in order for there to be no derivative loss in the transformation of u into v defined by $\mathcal{L}v = f(u,\cdots)$, a condition necessary to obtain a contraction mapping, the nonlinearity f can contain only u, u_x, u_t, u_{xx} and u_{xt}. We define the following normed function spaces:

<u>Definition A1</u>:

$$C = \{u(x,t) \,|\, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}, u_{xxt}, u_{xtt}, u_{xxxx}, u_{xxxt} \in C(\Omega)\} ,$$

where $C(\Omega)$ is the class of continuous functions on Ω.

Definition A2: B is the completion of $\mathcal{C}$ under the norm

$$\|u\| = |u|_m + |u_x|_m + |u_t|_m + |u_{xx}|_m + |u_{xt}|_m + |u_{tt}|_m$$
$$+ |u_{xxx}|_m + |u_{xxt}|_m + |u_{xxxx}|_{Lm} + |u_{xxxt}|_{Lm}$$
$$+ |u_{xtt}|_{Lm}$$

where $|\ |_m$ and $|\ |_{Lm}$ are defined in section 1.2, (17).

We also need an auxiliary **t-dependent norm**, $\|u\|_1$, defined as

$$\|u\|_1 = |u|_{m_1} + |u_x|_{m_1} + |u_t|_{m_1} + |u_{xx}|_{m_1}$$
$$+ |u_{xt}|_{m_1} + |u_{tt}|_{m_1} + |u_{xxx}|_{m_1} + |u_{xxt}|_{m_1} \qquad (A11)$$
$$+ |u_{xxxx}|_L + |u_{xxxt}|_L + |u_{xtt}|_L$$

where

$$|g(x,t)|_{m_1} = \max_{x\in[0,1]} |g(x,t)| ,$$

and

$$|g(x,t)|_L = \left(\int_0^1 g^2(x,t)dx\right)^{1/2} .$$

For a discussion of some of the properties of B see the Appendix.

We now state and prove two lemmas needed in the proofs of the existence and uniqueness theorems.

Lemma A1: Let $h(x,t) = \sum_1^\infty c_n(t)\varphi_n(x)$ where $c_n''(t)$ is continuous on $[0,T]$. Then $h \in B$ if the series $\sum_1^\infty c_n^2(t)n^8\pi^8$, $\sum_1^\infty c_n'^2(t)n^6\pi^6$ and $\sum_1^\infty c_n''^2(t)n^2\pi^2$ converge uniformly on $[0,T]$. Also

$$\|h\|_1 \leq 5\left[\left(\sum_1^\infty c_n^2(t)n^8\pi^8\right)^{1/2} + \left(\sum_1^\infty c_n'^2(t)n^6\pi^6\right)^{1/2} + \left(\sum_1^\infty c_n''^2(t)n^2\pi^2\right)^{1/2}\right] . \tag{A12}$$

Proof: Let $h_N = \sum_1^N c_n(t)\varphi_n(x)$; then by assumption $h_N \in C$. The uniform convergence of the three series insures that $\{h_N\}$ is a Cauchy sequence in B. The inequality (A12) follows from applications of the Schwarz inequality. For example, consider

$$h_{xxx} = \sum_1^\infty c_n(t)(-\sqrt{2}\, n^3\pi^3 \cos n\pi x)$$

and

$$h_{xxxx} = \sum_1^\infty c_n(t)n^4\pi^4\varphi_n(x) .$$

Therefore

$$h_{xxx}^2 \leq \left(\sum_1^\infty \frac{2\cos^2 n\pi x}{n^2\pi^2}\right)\left(\sum_1^\infty n^8\pi^8 c_n^2(t)\right) \leq \frac{1}{3}\sum_1^\infty n^8\pi^8 c_n^2(t)$$

and

$$|h_{xxxx}|_L = \left(\sum_1^\infty c_n^2(t)n^8\pi^8\right)^{1/2} .$$

Definition A3:

$$M = \{u \,|\, u \in B,\ u(x,0) = a_1(x),\ u_t(x,0) = a_2(x),\ u(0,t) = u(1,t) = 0\} .$$

Lemma A2: The linear nonhomogeneous problem $\mathcal{L}v = F(x,t)$, subject to (A2,3), has a solution $v \in M$ given by $v(x,t) = s_1 + s_2 + s_3$, where

$s_1 = \sum_1^\infty a_{1n}v_{1n}(t)\varphi_n(x)$, $s_2 = \sum_1^\infty a_{2n}v_{2n}(t)\varphi_n(x)$ and

$s_3 = \sum_1^\infty \int_0^t F_n(\nu)v_{2n}(t-\nu)d\nu\ \varphi_n(x)$. Moreover, there exist positive constants K_1 and K_2 depending on k (see (A9)-(A10)) such that

$$\|v\|_1 \leq K_1 e^{-at}(|a_{1xxxx}|_L + |a_{2xxx}|_L + |F_x(x,0)|_L) + K_2[(\int_0^t e^{-2a(t-\nu)} \int_0^1 F^2_{xx}(x,\nu)dxd\nu)^{1/2} + (\int_0^t e^{-2a(t-\nu)} \int_0^1 F^2_t(x,\nu)dx\, d\nu)^{1/2}] \tag{A13}$$

and

$$\|v\| \leq K_1(|a_{1xxxx}|_L + |a_{2xxx}|_L + |F_x(x,0)|_L) + K_2[(\int_0^T \int_0^1 F^2_{xx}(x,t)dxdt)^{1/2} + (\int_0^T \int_0^1 F^2_t(x,t)dxdt)^{1/2}] \tag{A14}$$

provided that

a. $F(x,t)$ is an indefinite integral in t for almost every x and $F_x(x,t)$ is an indefinite integral in x for every t ,

b. $F(0,t) = F(1,t) = 0$,

c. $\int_0^T \int_0^1 F^2_{xx}dxdt$ and $\int_0^T \int_0^1 F^2_t\, dxdt$ exist ,

d. $|a_{1xxxx}|_L$ and $|a_{2xxx}|_L$ exist ,

e. a_1, a_2, a_{1xx} and a_{2xx} vanish at $x = 0$ and $x = 1$.

Proof: Since B is a linear space $v \in B$ as long as s_1, s_2 and $s_3 \in B$. It is easy to verify that s_1 and $s_2 \in B$ and that

$$\|s_1\| + \|s_2\| \leq K_1 e^{-at}(|a_{1xxxx}|_L + |a_{2xxx}|_L)$$

by using conditions d) and e), the inequalities (A9), and Lemma A1. Therefore we focus our attention on s_3 and the corresponding $\sum_1^\infty C_n''^2 n^2\pi^2$ from Lemma A1.

We know $s_3 = \sum_1^\infty C_n(t)\varphi_n(x)$ where $C_n(t) = \int_0^t F_n(\nu)v_{2n}(t-\nu)d\nu$. Now if $F(x,t)$ is an indefinite integral in t for almost every x then $C_n''(t) = F_n(0)v_{2nt}(t) + \int_0^t F_n'(\nu)v_{2nt}(t-\nu)d\nu$. Since $\int_0^t F_n'(\nu)v_{2nt}(t-\nu)d\nu$ can be written as $\int_0^t e^{-a(t-\nu)}F_n'(\nu)e^{a(t-\nu)}\, v_{2nt}(t-\nu)d\nu$, an application of the Schwarz inequality and the use of (A9) and (A10) yield

$$C_n''^2(t) \leq 2k^2F_n^2(0)e^{-2at} + \frac{2k^2}{n^2\pi^2}\int_0^t e^{-2a(t-\nu)}F_n'^2(\nu)d\nu \,.$$

Parseval's equality, condition b and c and the monotone convergence theorem give us the following:

$$\sum_1^\infty n^2\pi^2F_n^2(0) = \int_0^1 F_x^2(x,0)dx$$

and

$$\begin{aligned}\sum_1^\infty \int_0^t e^{-2a(t-\nu)}F_n'^2(\nu)d\nu &= \int_0^t e^{-2a(t-\nu)} \sum_1^\infty F_n'^2(\nu)d\nu \\ &= \int_0^t e^{-2a(t-\nu)} \int_0^1 F_t^2(x,\nu)dx\, d\nu \\ &\leq \int_0^T \int_0^1 F_t^2(x,\nu)dx\, d\nu \,.\end{aligned}$$

Therefore $\sum_1^\infty C_n''^2(t)n^2\pi^2$ converges unformly and

$$\left(\sum_1^\infty C_n''^2(t)n^2\pi^2\right)^{1/2} \leq \Big(\sqrt{2}\, ke^{-at}|F_x(x,0)|_L$$
$$+ \sqrt{2}\, k\Big(\int_0^t e^{-2a(t-\nu)}\int_0^1 F_t^2(x,\nu)dx\, d\nu\Big)^{1/2} .$$

After similar calculations on $\sum_1^\infty C_n'^2 n^6\pi^6$ and $\sum_1^\infty C_n^2 n^8\pi^8$ are completed the fact that $v \in B$ and the inequalities (A13) and (A14) follow from Lemma A1. Since the series for v and v_t converge uniformly it is an easy matter to verify that $v(x,0) = a_1(x)$, $v_t(x,0) = a_2(x)$ and $v(0,t) = v(1,t) = 0$. Thus $v \in M$.

A good reference for the type of calculation used in the proof of this lemma is Weinberger [13].

2.2b. Existence and Uniqueness on $[0,T]$.

Definition A4: $M_1(\delta) = \{u | u \in M , \|u\| \leq \delta\}$.

Recall from definitions (A1)-(A3) that M depends on T.

Definition A5: $f(u,u_x,u_t,u_{xx},u_{xt},x,t) = F(x,t)$ is a Lipschitz function on $M_1(\delta)$ if there exist $L_1(\delta)$ and $L_2(\delta)$ such that

a. $|F_{xx}|_{Lm} + |F_t|_{Lm} \leq L_1(\delta)$ for every $u \in M_1$,

b. $|F_{1xx}-F_{2xx}|_{Lm} + |F_{1t}-F_{2t}|_{Lm} \leq L_2(\delta)\|u_1-u_2\|$ for every $u_1,u_2 \in M_1$ where

$$F_i(x,t) = f(u_i,u_{ix},u_{it},u_{ixx},u_{ixt},x,t) , \quad i = 1,2 .$$

In the Appendix we show that if $u \in B$ then u_{xt} is an indefinite integral in t for almost every x, u_{xxx} and u_{xxt} are indefinite integrals in x for every t and these derivatives exist in the Lm norm. It follows that F_{xx}, F_t and their Lm norms make sense. A sufficient condition for f to be Lipschitz on M_1 is that $f \in C^3$ on $[-\delta,\delta]^5 \times \Omega$.

We now have the machinery to prove the following

Theorem A1 (Existence and Uniqueness on a Finite Time Interval): Let η be an arbitrary number in $(0,1)$ and let f, a_1 and a_2 satisfy the following conditions:

a. f is Lipschitz on $M_1(\delta)$,

b. $F(0,t) = F(1,t) = 0$,

c. $K_1(|a_{1xxxx}|_L + |a_{2xxx}|_L + |F_x(x,0)|_L) \leq \eta\delta$,

d. a_1, a_2, a_{1xx} and a_{2xx} vanish at $x = 0$ and $x = 1$.

If $\sqrt{T} < \min\left(\frac{(1-\eta)\delta}{K_2 L_1}, \frac{1}{K_2 L_2}\right)$ then there exists a unique classical solution of (A1)-(A3) for $t \in [0,T]$. Recall that K_1 and K_2 are defined in Lemma A2.

Proof: To prove this it is sufficient to prove that A (see (15) and (A1)-(A7)) maps the complete metric space M_1 into itself and is a contraction on M_1.

Let $v = Au$; we want to prove that if $u \in M_1$ then $v \in M_1$. If we make the identification $F(x,t) = f(u,u_x,u_t,u_{xx},u_{xt},x,t)$ then conditions a-d of this theorem assure that the conditions of Lemma A2 are satisfied. It follows that $v \in M$ and that

$$\|v\| \le K_1(|a_{1xxxx}|_L + |a_{2xxx}|_L + |F_x(x,0)|_L)$$
$$+ K_2[(\int_0^T |F_{xx}|_L^2\, dt)^{1/2} + (\int_0^T |F_t|_L^2\, dt)^{1/2}]$$
$$\le \eta\delta + K_2 \sqrt{T}\, L_1(\delta)\ .$$

But $\sqrt{T} \le \frac{(1-\eta)\delta}{K_2 L_1}$ implies $\|v\| \le \delta$, therefore $v \in M_1$.

To prove that A is a contraction we must show there exists a positive $r < 1$ such that $\|Au_1 - Au_2\| \le r\|u_1 - u_2\|$ for every $u_1, u_2 \in M_1$. The inequality (A14) can be used here with $a_1 \equiv a_2 \equiv 0$ and F replaced by $F_1 - F_2$; it becomes:

$$\|Au_1 - Au_2\| \le K_2[(\int_0^T |F_{1xx} - F_{2xx}|_L^2\, dt)^{1/2} + (\int_0^T |F_{1t} - F_{2t}|_L^2\, dt)^{1/2}]$$

$$\le K_2 \sqrt{T}\, [|F_{1xx} - F_{2xx}|_{Lm} + |F_{1t} - F_{2t}|_{Lm}]\ .$$

Using a, $\|Au_1 - Au_2\| \le K_2 L_2(\delta)\sqrt{T}\, \|u_1 - u_2\|$. But $K_2 L_2(\delta)\sqrt{T} < 1$.

Theorem A1 follows by applying the contraction mapping theorem described in section 1.2.

2.2c. Existence and Uniqueness on $[0,\infty)$ and Asymptotic Stability.

Definition A6: $M_2(\delta,a) = \{u | u \in M \ , \ \ \|u\|_1 \le \delta e^{-at}\}$.

Lemma A3: $M_2(\delta,a)$ is complete.

Proof: Let $\{u_n\}$ be a Cauchy sequence in $M_2 \subset M$. Since M is complete there exists a $u \in M$ such that $u_n \to u$. The Lemma is proved if $\|u\|_1 \le \delta e^{-at}$.

From the triangle inequality

$$\|u\|_1 \le \|u-u_n\|_1 + \|u_n\|_1 \le \|u-u_n\|_1 + \delta e^{-at} \ ,$$

for n large

$$\|u\|_1 \le \epsilon + \delta e^{-at}$$

for every $\epsilon > 0$. Therefore $\|u\|_1 \le \delta e^{-at}$.

Definition A7: $f(u,u_x,u_t,u_{xx},u_{xt},x,t) = F(x,t)$ is an exponential Lipschitz function on $M_2(\delta,a)$ if there exist $L_1(\delta)$ and $L_2(\delta)$ such that

a. $|F_{xx}|_L$ and $|F_t|_L \le \frac{1}{2} L_1\|u\|_1^2 \le \frac{1}{2} L_1 \|u\|_1 \ \delta e^{-at}$ for every $u \in M_2$,

b. $|F_{1xx}-F_{2xx}|_L$ and $|F_{1t}-F_{2t}|_L \le \frac{1}{2} L_2\|u_1-u_2\|\delta$ for every $u_1,u_2 \in M_2$ where

$$F_i(x,t) = f(u_i,u_{ix},u_{it},u_{ixx},u_{ixt},x,t) \ , \quad i = 1,2 \ .$$

Theorem A2 (Existence and Uniqueness for every $t > 0$ and Asymptotic Stability): Let η be an arbitrary number in $(0,1)$ and let f, a_1 and a_2 satisfy the following conditions:

a. f is exponential Lipschitz on $M_2(\delta,a)$,

b. $F(0,t) = F(1,t) = 0$,

c. $K_1(|a_{1xxxx}|_L + |a_{2xxx}|_L + |F_x(x,0)|_L) \leq \eta\delta$,

d. a_1, a_2, a_{1xx} and a_{2xx} vanish at $x = 0$ and $x = 1$.

If δ is such that the inequalities

$$L_1(\delta)\delta \leq (1-\eta)(2a)^{1/2} /K_2$$

and

$$L_2(\delta)\delta < (2a)^{1/2} /K_2$$

are satisfied, then there exists a unique classical solution of (A1)-(A3) in M_2 for every $T > 0$.

Remarks:

a. The exponential Lipschitz condition implies that $u \equiv 0$ is an equilibrium solution. The nature of M_2 insures the asymptotic stability of the zero solution.

b. Condition a of Theorem A2 can be weakened; however, for expository purposes we feel the above statement of the theorem is the most instructive. For example, a form of Theorem A2 can be proved under the condition that $|F_{xx}|_L$ and $|F_t|_L \leq \frac{1}{2} L_1(\delta)\|u\|_1^{1+\epsilon}$ for $\epsilon > 0$.

Proof: To prove the theorem it is sufficient to prove that A maps the complete metric space M_2 into itself and is a contraction on M_2.

Let $v = Au$; we want to prove that if $u \in M_2$ then $v \in M_2$. It follows from conditions a through d and Lemma A2 that $v \in M$ and that

$$\|v\|_1 \le K_1 e^{-at}(|a_{1xxxx}|_L + |a_{2xxx}|_L + |F_x(x,0)|_L)$$
$$+ K_2[(\int_0^t e^{-2a(t-\nu)}|F_{xx}|_L^2\, d\nu)^{1/2} + (\int_0^t e^{-2a(t-\nu)}|F_t|_L^2\, d\nu)^{1/2}]$$
$$\le \eta\delta\, e^{-at} + K_2 L_1 \delta^2 (2a)^{-1/2} e^{-at} .$$

But $L_1(\delta)\delta \le (1-\eta)(2a)^{1/2}/K_2$ implies $\|v\|_1 \le \delta e^{-at}$, therefore $v \in M_2$.

To prove that A is a contraction we must show there exists a positive $r < 1$ such that $\|Au_1 - Au_2\| \le r\|u_1 - u_2\|$ for every $u_1, u_2 \in M_2$. The inequality (A13) can be used here with $a_1 \equiv a_2 \equiv 0$ and F replaced by $F_1 - F_2$; it becomes:

$$\|Au_1 - Au_2\|_1 \le K_2[(\int_0^t e^{-2a(t-\nu)}|F_{1xx} - F_{2xx}|_L^2\, d\nu)^{1/2}$$
$$+ (\int_0^t e^{-2a(t-\nu)}|F_{1t} - F_{2t}|_L^2\, d\nu)^{1/2}] .$$

Condition a and the inequality $L_2(\delta)\delta < (2a)^{1/2}/K_2$ imply

$$\|Au_1 - Au_2\|_1 \le K_2 L_2 \delta (2a)^{-1/2}\|u_1 - u_2\| < r\|u_1 - u_2\|$$

with $r < 1$. Therefore

$$\|Au_1 - Au_2\| \le r\|u_1 - u_2\| .$$

Theorem A2 follows by applying the contraction mapping theorem described in section 1.2 and noting that there is no restriction on T in the proof of the theorem.

<u>Theorem A3</u>: Suppose all the conditions of Theorem A2 are satisfied, then

$$\|u\|_1 \leq \sqrt{2}\ \eta\delta\ \exp[(1-\eta)^2(1-e^{-2at})]e^{-at}$$

where u is the fixed point of (A1)-(A3).

<u>Proof</u>: Lemma A2, condition a of Theorem A2 and the fact that u is a fixed point of (A1)-(A3) imply that

$$\|u\|_1 \leq K_1 e^{-at}(|a_{1xxxx}|_L + |a_{2xxx}|_L + |F_x(x,0)|_L)$$
$$+ K_2 L_1 \left(\int_0^t e^{-2a(t-\nu)} \|u\|_1^4 \, d\nu \right)^{1/2} .$$

Condition c of Theorem A2 , the fact that $\|u\|_1 \leq \delta e^{-at}$ and the fact that $L_1\delta \leq (1-\eta)(2a)^{1/2}/K_2$ imply that

$$e^{at}\|u\|_1 \leq \eta\delta + (1-\eta)(2a)^{1/2}\left(\int_0^t \|u\|_1^2 \, d\nu \right)^{1/2} .$$

Let $y(t) = e^{at}\|u\|_1$, then

$$y(t) \leq \eta\delta + (1-\eta)(2a)^{1/2}\left(\int_0^t e^{-2a\nu} y^2 d\nu \right)^{1/2} . \tag{A15}$$

If we square both sides and use the fact that $(p+q)^2 \leq 2p^2 + 2q^2$ we find

$$y^2(t) \leq 2\eta^2\delta^2 + 4a(1-\eta)^2 \int_0^t e^{-2a\nu} y^2 d\nu . \tag{A16}$$

An application of Gronwell's inequality to (A16) gives the desired result:

$$\|u\|_1^2 \, e^{2at} = y^2 \leq 2\eta^2\delta^2 \exp[2(1-\eta)^2(1-e^{-2at})]$$

or

$$\|u\|_1 \leq \sqrt{2}\ \eta\delta\ \exp[(1-\eta)^2(1-e^{-2at})]e^{-at} .$$

Remark: The step (A15) to (A16) is unfortunate as (A15) gives a better bound than (A16); however, it appears to be necessary in order to get a closed form solution of the inequality. If the approach used in the derivation of Gronwall's inequality is attempted on (A15) a differential inequality is derived which cannot be solved in closed form.

2.2d. Construction of a Liapunov Functional for Special f's .

We consider two specific types of nonlinearities f, $f = -g(u)$ and $f = \frac{\partial}{\partial x} g(u_x)$ where g has the properties

$$zg(z) > 0 \quad z \neq 0 \tag{A17}$$

$$g'(z) > 0 . \tag{A18}$$

The Liapunov functional for each type of nonlinearity will be of the form $V(t) = V_1(t) + V_2(t)$ where V_1 is defined in (24)

$$V_1(t) = \frac{1}{2}\int_0^1 [u_x^2 + (u_t-\alpha u_{xx})^2 + \alpha^2 u_{xx}^2]dx . \tag{A19}$$

From (25)

$$\dot{V}_1(t) = -\alpha \int_0^1 [u_{xt}^2 + u_{xx}^2 + u_{xx}f]dx + \int_0^1 u_t f \, dx . \tag{A20}$$

Notice that $V_1(t)$ is positive definite and that for the f's we are considering $\int_0^1 u_{xx}fdx$ is nonnegative. The basic idea in constructing $V(t)$ is to find a $V_2(t)$ which is nonnegative such that $\dot{V}_2(t)$ cancels out the last term in (A20). It must be remembered that u satisfies (A2).

Theorem A4: Let $f = -g(u)$ where g satisfies (A17)-(A18). Then $V(t) = V_1(t) + V_2(t)$ where

$$V_2(t) = \int_0^1 \left(\int_0^{u(x,t)} g(\xi)d\xi\right)dx \tag{A21}$$

is a Liapunov functional for (A1)-(A2). It follows from the Liapunov stability theorem that the zero solution of (A1)-(A2) is asymptotically stable in the large.

Proof: Differentiation of (A21) yields

$$\dot{V}_2(t) = \int_0^1 u_t g(u)dx ,$$

which leads to

$$V(t) = \frac{1}{2}\int_0^1 \{u_x^2 + (u_t-\alpha u_{xx})^2 + \alpha^2 u_{xx}^2 + 2\int_0^u g(\xi)d\xi\}dx \tag{A22}$$

and

$$\dot{V}(t) = -\alpha \int_0^1 (u_{xt}^2 + u_{xx}^2 + u_x^2\, g'(u))dx . \tag{A23}$$

The theorem follows once we show (A22) and (A23) satisfy conditions a) through d) of the Liapunov stability theorem with

$$\rho^2(t) = \int_0^1 [u^2 + u_x^2 + u_t^2 + u_{xx}^2]dx . \tag{A24}$$

The following inequalities will be useful in our proof. Since u is such that $u(0,t) = u(1,t) = 0$ and u_{xx} exists,

$$|u(x,t)| \le \int_0^1 |u_x(x,t)|dx \Rightarrow u^2 \le \int_0^1 u_x^2\, dx \le \rho^2 \tag{A25}$$

and

$$|u_x(x,t)| \le \int_0^1 |u_{xx}(x,t)|dx \Rightarrow u_x^2 \le \int_0^1 u_{xx}^2\, dx \le \rho^2 . \tag{A26}$$

Inequality (A26) is based on the fact that for each t , $u_x(x,t)$ has a zero for some x (by Rolle's theorem).

To verify conditions a and d of the Liapunov stability theorem we make use of (A25) and the inequality $|u_t| \leq |u_t-\alpha u_{xx}| + \alpha|u_{xx}|$ to find

$$\rho^2 \leq \int_0^1 [2u_x^2 + 2(u_t-\alpha u_{xx})^2 + 2\alpha^2 u_{xx}^2 + u_{xx}^2]\, dx$$

$$\leq 2(2 + {}^1/\alpha^2)V_1(t) = K_1 V_1(t) \leq K_1 V(t) \; .$$

Therefore $V(t) \geq \frac{\rho^2}{K_1} \equiv \beta_1(\rho)$ and clearly β_1 satisfies the appropriate conditions.

To verify condition c we notice that there exists a K_2 such that

$$V_1(t) \leq \frac{1}{2} \int_0^1 [u_x^2 + 2u_t^2 + 3\alpha^2 u_{xx}^2]dx \;\leq\; K_2\rho^2 \; .$$

Now consider $V_2(t) = \int_0^1 (\int_0^u g(\xi)d\xi)dx$ and let $h(|u|) \equiv \max(\int_0^u g(\xi)d\xi \, , \, \int_0^{-u} g(\xi)d\xi)$. This implies that $h(0) = 0$. From (A17) we see h is a nondecreasing function of $|u|$ and $\int_0^u g(\xi)d\xi \leq h(|u|)$. By (A25) $|u| \leq \rho$, therefore,

$$V_2(t) = \int_0^1 \int_0^u g(\xi)d\xi \, dx \leq \int_0^1 h(\rho)dx = h(\rho) \; .$$

Combining this with the result for V_1 yields $V(t) \leq K_2\rho^2 + h(\rho) \equiv \beta_2(\rho)$ and clearly β_2 satisfies the appropriate conditions.

Condition b will be satisfied if we find a γ such that $\gamma(\rho) \leq \alpha \int_0^1 [u_{xt}^2 + u_{xx}^2 + u_x^2 g'(u)]dx$. Using (A25) and (A26) we find

$$\rho^2 = \int_0^1 [u^2 + u_x^2 + u_t^2 + u_{xx}^2]dx \leq \int_0^1 [3u_{xx}^2 + u_{xt}^2]dx ,$$

but $g'(u)$ is positive; therefore,

$$\rho^2 \leq 3 \int_0^1 [u_{xx}^2 + u_{xt}^2 + u_x^2 \, g'(u)]dx .$$

It we take $\gamma(\rho) = \frac{\alpha\rho^2}{3}$, condition b is satisfied.

This completes the proof of Theorem A4.

Theorem A5: Let $f = \frac{\partial}{\partial x} g(u_x)$ where g satisfies (A17)-(A18). Then $V(t) = V_1(t) + V_2(t)$ where

$$V_2(t) = \int_0^1 \left(\int_0^{u_x(x,t)} g(\xi)d\xi \right)dx \tag{A27}$$

is a Liapunov functional for (A1)-(A2). It follows from the Liapunov stability theorem that the zero solution of (A1)-(A2) is asymptotically stable in the large.

Proof: The proof of this theorem follows the proof of Theorem A4, with the appropriate modifications in the construction of $\beta_2(\rho)$.

Remark: It is interesting to note that in the case where $f \equiv 0$, $V_1(t)$ is a Liapunov functional and there exists a positive constant c such that $V_1(t) \leq V_1(0)e^{-\alpha t/c}$. So, not only do we have asymptotic stability in the large but we have an estimate on the rate of decay of the Liapunov functional. Similar bounds can be obtained in theorems A4 and A5 by making further restrictions on g.

2.3 CASE B.

2.3a. Preliminaries.

The differential equation is

$$u_{tt} + 2\alpha u_t - u_{xx} = f(u, u_x, u_t, x, t)$$
$$x \in [0,1], \quad t \geq 0, \quad \alpha > 0, \tag{B1}$$

with boundary and initial conditions

$$u(0,t) = u(1,t) = 0, \tag{B2}$$

$$u(x,0) = a_1(x), \quad u_t(x,0) = a_2(x). \tag{B3}$$

We shall give neither the lemmas nor proofs of the theorems which are parallel to those in Case A; an understanding of Case A should be sufficient for filling these in.

<u>Definition B1</u>: $\quad C = \{u(x,t) \mid u \in C^3(\Omega)\}$

where $\Omega = [0,1] \times [0,T]$ and C^3 denotes the class of 3 times continuously differentiable functions.

<u>Definition B2</u>: B is the completion of C under the norm

$$\|u\| = \sum_{|\sigma|=0}^{2} |D^\sigma u|_m + \sum_{|\sigma|=3} |D^\sigma u|_{Lm} \tag{B4}$$

where

$$\sigma = (\sigma_1, \sigma_2), \quad |\sigma| = \sigma_1 + \sigma_2, \; D^\sigma = \frac{\partial^{\sigma_1+\sigma_2}}{\partial x^{\sigma_1} \partial t^{\sigma_2}}$$

and $|\;|_m$ and $|\;|_{Lm}$ are the same as in Case A.

The properties of B can be discussed in exactly the same way as in the Appendix.

<u>Definition B3</u>: $\quad M = \{u \mid u \in B,\; u(x,0) = a_1(x),\; u_t(x,0) = a_2(x),$
$u(0,t) = u(1,t) = 0\}$.

2.3b. Existence and Uniqueness on [0,T].

<u>Definition B4</u>: $M_1(\delta) = \{u \mid u \in M\ ,\ \|u\| \leq \delta\}$.

<u>Definition B5</u>: $f(u,u_x,u_t,x,t) = F(x,t)$ is a Lipschitz function on $M_1(\delta)$ if there exist $L_1(\delta)$ and $L_2(\delta)$ such that

a. $|F_{xx}|_{Lm} + |F_{tt}|_{Lm} \leq L_1(\delta)$ for every $u \in M_1$,

b. $|F_{1xx}-F_{2xx}|_{Lm} + |F_{1tt} - F_{2tt}|_{Lm} \leq L_2(\delta)\|u_1-u_2\|$ for every $u_1,u_2 \in M_1$ where

$$F_i(x,t) = f(u_i,u_{ix},u_{it},x,t), \quad i = 1,2 .$$

<u>Theorem B1</u> (<u>Existence and Uniqueness on a Finite Time Interval</u>): Let η be an arbitrary number in (0,1) and let f, a_1 and a_2 satisfy the following conditions:

a. f is Lipschitz on $M_1(\delta)$,

b. $F(0,t) = F(1,t) = 0$.

c. $K_1(|a_{1xxx}|_L + |a_{2xx}|_L + |F_x(x,0)|_L + |F_t(x,0)|_L) \leq \eta\delta$,

d. a_1, a_{1xx} and a_2 vanish at $x = 0$ and $x = 1$.

If $\sqrt{T} < \min\left(\frac{(1-\eta)\delta}{K_2L_1}\ ,\ \frac{1}{K_2L_2}\right)$ then there exists a unique classical solution of (B1)-(B3) for $t \in [0,T]$.

Remark: K_1 and K_2 are positive constants which can be determined as in Lemma A2.

2.3c. Existence and Uniqueness on $[0,\infty)$ and Asymptotic Stability.

<u>Definition B6</u>: $M_2(\delta,a) = \{u \mid u \in M\ ,\ \|u\|_1 \leq \delta e^{-at}\}$,

where

$$\|u\|_1 = \sum_{|\sigma|=0}^{2} |D^\sigma u|_{m_1} + \sum_{|\sigma|=3} |D^\sigma u|_L \tag{B5}$$

and $|\ |_{m_1}$ and $|\ |_L$ are the same as in Case A.

Definition B7: $f(u,u_x,u_t,x,t)$ is an exponential Lipschitz function on $M_2(\delta,a)$ if there exist $L_1(\delta)$ and $L_2(\delta)$ such that

a. $|F_{xx}|_L$ and $|F_{tt}|_L \leq \frac{1}{2} L_1\|u\|_1^2 \leq \frac{1}{2} L_1 \|u\|_1 \delta e^{-at}$ for every $u \in M_2$,

b. $|F_{1xx}-F_{2xx}|_L$ and $|F_{1tt}-F_{2tt}|_L \leq \frac{1}{2} L_2 \|u_1-u_2\|\delta$ for every $u_1,u_2 \in M_2$, where F_1 and F_2 are as in definition B5.

Theorem B2 (Existence and Uniqueness for every $t \geq 0$ and Asymptotic Stability): Let η be an arbitrary number in $(0,1)$ and let f, a_1 and a_2 satisfy the following conditions:

a. f is exponential Lipschitz on $M_2(\delta,a)$,

b. $F(0,t) = F(1,t) = 0$,

c. $K_1(|a_{1xxx}|_L + |a_{2xx}|_L + |F_x(x,0)|_L + |F_t(x,0)|_L) \leq \eta\delta$,

d. a_1, a_{1xx} and a_2 vanish at $x = 0$ and $x = 1$.

If δ is such that the inequalities

$$L_1(\delta)\delta \leq (1-\eta)(2a)^{1/2} /K_2$$

and

$$L_2(\delta)\delta \leq (2a)^{1/2} /K_2$$

are satisfied then there exists a unique classical solution of (B1)-(B3) in M_2 for every $T > 0$.

The remarks made after Theorem A2 are appropriate here.

Theorem B3: Suppose all the conditions of Theorem B2 are satisfied, then the inequality of Theorem A3 holds where u is the fixed point of (B1)-(B3).

2.3d. Liapunov Functional for Stability for Special f's .

We consider the same two specific types of f as in Case A, but let

$$\rho^2 = \int_0^1 [u^2 + u_x^2 + u_t^2]dx \ .$$

Theorem B4: Let $f = -g(u)$. Then

$$V(t) = \frac{1}{2}\int_0^1 [u_x^2 + (u_t+\alpha u)^2 + \alpha^2u^2 + 2\int_0^u g(\xi)d\xi]dx \qquad \text{(B6)}$$

is a Liapunov functional for (B1)-(B2), where

$$\dot{V}(t) = -\alpha\int_0^1 [u_t^2 + u_x^2 + u\, g(u)]dx \ .$$

Theorem B5: Let $f = \frac{\partial}{\partial x} g(u_x)$. Then

$$V(t) = \frac{1}{2}\int_0^1 [u_x^2 + (u_t+ \alpha u)^2 + \alpha^2u^2 + 2\int_0^{u_x} g(\xi)d\xi]dx \qquad \text{(B7)}$$

is a Liapunov functional for (B1)-(B2) .

(Notice the existence theorems do not apply to nonlinearities of this type.)

The remark after Theorem A5 is appropriate here.

2.4 CASE C.

2.4a. Preliminaries.

The differential equation is

$$\mathcal{L}u \equiv u_t - u_{xx} = f(u,u_x,x,t) \qquad x \in [0,1], \quad t \geq 0 , \tag{C1}$$

with boundary and initial conditions

$$u(0,t) = u(1,t) = 0 , \tag{C2}$$

$$u(x,0) = a_1(x) . \tag{C3}$$

We discuss this case in the same way as Case B except we briefly consider the mapping since it differs from (15).

The mapping of u into v defined by

$$v_t - v_{xx} = f(u(x,t),u_x(x,t),x,t) \equiv F(x,t) \tag{C4}$$

can be written in a form similar to (15) as

$$v = Au = \sum_1^{\infty} a_{1n}\, v_{1n}(t)\varphi_n(x) + \sum_1^{\infty} \left(\int_0^t F_n(\nu)v_{1n}(t-\nu)d\nu\right)\varphi_n(x) \tag{C5}$$

where

$$\left.\begin{aligned} v_{1n}(t) &= e^{-n^2\pi^2 t} \\ \varphi_n(x) &= \sqrt{2}\,\sin n\pi\, x \\ a_{1n} &= \int_0^1 a_1(x)\varphi_n(x)dx \\ &\text{and} \\ F_n(t) &= \int_0^1 F(x,t)\varphi_n(x)dx . \end{aligned}\right\} \tag{C6}$$

The solution of (C1)-(C3) can be viewed as a fixed point of the mapping A in (C5).

Definition C1: $C = \{u | u, u_x, u_t, u_{xx}, u_{xxx}, u_{xt} \in C(\Omega)\}$ where $\Omega = [0,1] \times [0,T]$.

Definition C2: B is the completion of C under the norm

$$\|u\| = |u|_m + |u_x|_m + |u_t|_m + |u_{xx}|_m + |u_{xt}|_{Lm} + |u_{xxx}|_{Lm}$$

where $|\ |_m$ and $|\ |_{Lm}$ are the same as in case A.

The properties of B can be discussed in exactly the same way as in the Appendix.

Definition C3: $M = \{u | u \in B,\ u(x,0) = a_1(x),\ u(0,t) = u(1,t) = 0\}$.

2.4b. Existence and Uniqueness on $[0,T]$.

Definition C4: $M_1(\delta) = \{u | u \in M,\ \|u\| \leq \delta\}$.

Definition C5: $f(u,u_x,x,t) = F(x,t)$ is Lipschitz on $M_1(\delta)$ if there exist $L_1(\delta)$ and $L_2(\delta)$ such that

a. $|F_{xx}|_{Lm} + |F_t|_{Lm} \leq L_1(\delta)$ for every $u \in M_1$,

b. $|F_{1xx} - F_{2xx}|_{Lm} + |F_{1t} - F_{2t}|_{Lm} \leq L_2(\delta)\|u_1 - u_2\|$ for every $u_1, u_2 \in M_1$ where

$$F_i(x,t) = f(u_i, u_{ix}, x, t), \quad i = 1,2 .$$

Theorem C1 (Existence and Uniqueness on a Finite Time Interval): Let η be an arbitrary number in $(0,1)$ and let f and $a_1(x)$ satisfy the following conditions:

a. f is Lipschitz on $M_1(\delta)$,

b. $F(0,t) = F(1,t) = 0$,

c. $K_1(|a_{1xxx}|_L + |F_x(x,0)|_L) \leq \eta\delta$,

d. a_1 and a_{1xx} vanish at $x = 0$ and $x = 1$.

If $\sqrt{T} < \min\left(\frac{(1-\eta)\delta}{K_2L_1}, \frac{1}{K_2L_2}\right)$ then there exists a unique classical solution of (C1)-(C3) for $t \in [0,T]$.

Remark: K_1 and K_2 are positive constants which can be determined as in Lemma A2.

2.4c. Existence and Uniqueness on $[0,\infty)$ and Asymptotic Stability.

Definition C6: $M_2(\delta,a) = \{u|u \in M, \|u\|_1 \leq \delta e^{-at}\}$, where $\|u\|_1 = |u|_{m_1} + |u_x|_{m_1} + |u_t|_{m_1} + |u_{xx}|_{m_1} + |u_{xt}|_L + |u_{xxx}|_L$, and $|\ |_{m_1}$ and $|\ |_L$ are as in Case A.

Definition C7: $f(u,u_x,x,t) = F(x,t)$ is an exponential Lipschitz function on M_2 if there exist $L_1(\delta)$ and $L_2(\delta)$ such that

a. $|F_{xx}|_L$ and $|F_t|_L \leq \frac{1}{2} L_1 \|u\|_1^2 \leq \frac{1}{2} L_1 \|u\|_1 \delta e^{-at}$ for every $u_1,u_2 \in M_2$,

b. $|F_{1xx} - F_{2xx}|_L$ and $|F_{1t} - F_{2t}|_L \leq \frac{1}{2} L_2 \|u_1-u_2\|\delta$ for every $u_1,u_2 \in M_2$, where F_1 and F_2 are as in definition C5 .

Theorem C2 (Existence and Uniqueness for every $t \geq 0$ and Asymptotic Stability): Let η be an arbitrary number in $(0,1)$ and let f and a_1 satisfy the following conditions:

a. f is exponential Lipschitz on $M_2(\delta,a)$,

b. $F(0,t) = F(1,t) = 0$,

c. $K_1(|a_{1xxx}|_L + |F_x(x,0)|_L) \leq \eta\delta$,

d. a_1 and a_{1xx} vanish at $x = 0$ and $x = 1$.

If δ is such that the inequalities

$$L_1(\delta)\delta \leq (1-\eta)(2a)^{1/2}/K_2$$

and

$$L_2(\delta)\delta < (2a)^{1/2} /K_2$$

are satisfied, then there exists a unique classical solution of (C1)-(C3) in M_2 for every $T > 0$.

The remarks made after Theorem A2 are appropriate here.

Theorem C3: Suppose all the conditions of Theorem C2 are satisfied, then the inequality of Theorem A3 holds for Case C where u is the fixed point of (C1)-(C3).

2.4d. Liapunov Functional for Stability for Special f's .

We consider the same two specific types of f as in case A, but let

$$\rho^2 = \int_0^1 (u^2 + u_x^2)dx .$$

Theorem C4:

$$V(t) = \frac{1}{2}\int_0^1 (u^2 + u_x^2)dx$$

is a Liapunov functional for (C1)-(C2) for both nonlinearities.

2.5 CASE D.

The differential equation and auxiliary conditions are:

$$\mathcal{L}u \equiv u_{tt} - 2\alpha \nabla^2 u_t - \nabla^2 u = f(u,u_x,u_y,u_t,x,y,t)$$
$$x,y \in \mathcal{D}\ , \quad t \geq 0\ , \quad \alpha > 0 \tag{D1}$$

where $\mathcal{D} = [0,1]^2$,

$$u(x,y,t) = 0 \quad \text{on} \quad \partial\mathcal{D} \tag{D2}$$

where $\partial\mathcal{D}$ is the boundary of $\mathcal{D}$, and

$$u(x,y,0) = a_1(x,y), \quad u_t(x,y,0) = a_2(x,y)\ . \tag{D3}$$

Results for this case can be obtained just as in the other cases, but because of space limitations we shall just indicate some points which may not be obvious simply from a study of the previous cases. This case was selected to illustrate that the ideas in this paper are applicable to problems containing more than one spatial dimension.

1. Definitions of C and B .

$C = \{u | D^\sigma u \in C(\Omega),\ \ 0 \leq |\sigma| \leq 4\ ;\ D^\sigma u_t \in C(\Omega),\ \ 0 \leq |\sigma| \leq 3\ ;$
$D^\sigma u_{tt} \in C(\Omega),\ \ 0 \leq |\sigma| \leq 1\}$ where

$$\sigma = (\sigma_1,\sigma_2), \quad |\sigma| = \sigma_1 + \sigma_2\ , \quad D^\sigma = \frac{\partial^{\sigma_1+\sigma_2}}{\partial x^{\sigma_1}\ \partial y^{\sigma_2}}$$

and $\Omega = \mathcal{D} \times [0,T]$.

B is the completion of C under the norm

$$\|u\| = \sum_{|\sigma|=0}^{3} |D^{\sigma}u|_m + \sum_{|\sigma|=0}^{2} |D^{\sigma}u_t|_m + |u_{tt}|_m$$

$$+ \sum_{|\sigma|=4} |D^{\sigma}u|_{Lm} + \sum_{|\sigma|=3} |D^{\sigma}u_t|_{Lm} + |u_{xtt}|_{Lm} + |u_{ytt}|_{Lm}$$

where

$$|g(x,y,t)|_m = \max_{\Omega} |g(x,y,t)|$$

and

$$|g(x,y,t)|_{Lm} = \max_{t\in[0,T]} \left(\int_{\mathscr{D}} g^2(x,y,t)dx\,dy\right)^{1/2} .$$

The modifications needed inthe Appendix in order to discuss the properties of the space B in this case are not as trivial as in Cases B and C.

2. The Lemma corresponding to Lemma A1 is not as good in this case because of the divergence of the series $\sum_1^{\infty}\sum_1^{\infty} \frac{1}{m^2+n^2}$. (Notice the role of the convergence of $\sum_1^{\infty} \frac{1}{n^2}$ in the proof of Lemma A1.) The result of this is that the nonlinearity in (D1) can contain the derivatives u_x, u_y and u_t but not higher order ones.

3. The Lemma corresponding to Lemma A2 requires smoothness conditions on $\nabla^2 F$ and F_t (compare a. of Lemma A2) and the integrability of $|\nabla(\nabla^2 F)|$ and $|\nabla F_t|$ (compare c. of Lemma A2) .

With the understanding of 1 - 3 , the proofs of existence, and uniqueness go through much the same as in the other cases.

4. A Liapunov functional for $f = -g(u)$ subject to the conditions (A17-18) is

$$V(t) = \frac{1}{2}\int_{\mathscr{D}} [u_x^2+u_y^2+(u_t-\alpha\nabla^2 u)^2+\alpha^2(\nabla^2 u)^2 + \int_0^u g(\xi)d\xi]dx\,dy .$$

APPENDIX

PROPERTIES OF SPACE B IN CASE A

Recall the definition of B (A1 and A2) and notice the structure of the norm defined there. It is clear that the continuity of the derivatives which are completed under the $|\ |_m$ norm is preserved, i.e., if $u \in B$ then $u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}$ and u_{xxt} are continuous on $\Omega = [0,1] \times [0,T]$. The question we consider here is: what happens to the x derivatives of u_{tt}, u_{xx} and u_{xxt} and the t derivatives of u_{xt} and u_{xx} under the $|\ |_{Lm}$ norm in the completion of $C(\Omega)$?

Consider two spaces B_1 and B_2 where B_1 is the completion of $C_1(\Omega) = \{w(x,t) | w, w_x \in C(\Omega)\}$ under the norm $\|w\| = |w|_m + |w_x|_{Lm}$ and B_2 is the completion of $C_2(\Omega) = \{w(x,t) | w, w_t \in C(\Omega)\}$ under the norm $\|w\| = |w|_m + |w_t|_{Lm}$. The properties of the x derivatives of u_{tt}, u_{xxx} and u_{xxt} and the t derivatives of u_{xt} and u_{xxx} follow from the properties of w in B_1 and B_2 respectively.

To discuss the properties of B_1 and B_2 we need

<u>Lemma 1:</u> Let $\{v_n\}$ be a Cauchy sequence in $C(\Omega)$ under the norm $|\ |_{Lm}$ (i.e. $|v_n - v_m|_{Lm} \to 0$). Then there exists an $f(x,t)$ such that

a. $\lim\limits_n |v_n - f|_L = 0$ for each $t \in [0,T]$,

b. $|f|_{Lm}$ exists and $\lim\limits_n |v_n|_{Lm} = |f|_{Lm}$

and

c. $\lim\limits_n |v_n - f|_{Lm} = 0$.

<u>Proof</u>: Since $|v_n - v_m|_L \leq |v_n - v_m|_{Lm}$, $\{v_n\}$ is a Cauchy sequence in $L_2[0,1]$ for every $t \in [0,T]$. The Riesz-Fischer theorem (see Korevaar [8], page 391) asserts the existence of an $f(x,t) \in L_2[0,1]$ for each $t \in [0,T]$ such that $\lim\limits_n |v_n - f|_{Lm} = 0$.

Since $f(x,t) \in L_2[0,1]$, $|f|_L$ exists for every $t \in [0,T]$ and the existence of $|f|_{Lm}$ follows if $|f|_L$ is a continuous function of t. To prove this we show that $\lim_n |v_n|_L = |f|_L$ uniformly in t for $t \in [0,T]$: The result $||v_n|_L - |f|_L| \leq |v_n - f|_L \to 0$ implies $\lim_n |v_n|_L = |f|_L$ pointwise in t. Let $h_n(t) = |v_n|_L$ then

$$|h_n(t) - h_m(t)| \leq |v_n - v_m|_L \leq |v_n - v_m|_{Lm} \to 0 \ .$$

Therefore $|f|_L$ is the uniform limit of a sequence of continuous functions which assures the continuity of $|f|_L$ and the existence of $|f|_{Lm}$.

To prove c) we notice that for every $\varepsilon > 0$ there exists an n such that

$$|v_{n+p} - v_n|_{Lm} \leq \frac{1}{2}\varepsilon \quad \text{for every} \quad p \geq 0 \ .$$

But

$$\lim_p |v_{n+p} - v_n|_{Lm} = |f - v_n|_{Lm} \leq \frac{1}{2}\varepsilon < \varepsilon$$

therefore

$$\lim_n |v_n - f|_{Lm} = 0 \ .$$

Consider the spaces B_1 and B_2.

Lemma 2 (Properties of B_1): Let $w \in B_1$. Then w is an indefinite integral in x for every t.

Lemma 3 (Properties of B_2): Let $w \in B_2$. Then w is an indefinite integral in t for almost every x.

Proof of Lemma 2: For every w in B_1 there exists an associated Cauchy sequence $\{w_n\}$, $w_n \in C_1$ such that

$$\begin{array}{c} w_{nx} \text{ is a Cauchy sequence in } C(\Omega) \\ \text{under the norm } |\;|_{Lm} \end{array} \qquad \text{(i)}$$

and

$$w_n(x,t) = w_n(0,t) + \int_0^x w_{nx}(\nu,t)d\nu \ . \tag{ii}$$

We know from Lemma 1 and (i) that there exists a $g(x,t)$ such that $\lim_n |w_{nx}-g|_{Lm} = 0$. The equality

$$\lim_n \int_0^x w_{nx}(\nu,t)d\nu = \int_0^x g(\nu,t)d\nu \tag{iii}$$

for every $x \in [0,1]$ and for every $t \in [0,T]$, follows from the inequalities

$$\left| \int_0^x (w_{nx}(\nu,t)-g(\nu,t))d\nu \right| \le \int_0^1 |w_{nx}(\nu,t)-g(\nu,t)|d\nu$$

$$\le |w_{nx}-g|_L \le |w_{nx}-g|_{Lm} \rightarrow 0 \ .$$

If we make use of (iii) and the uniform convergence of $\{w_n(x,t)\}$, the pointwise limit of equation (ii) yields

$$w(x,t) = w(0,t) + \int_0^x g(\nu,t)d\nu \ . \tag{iv}$$

Therefore $w(x,t)$ is an indefinite integral in x for every t.

<u>Proof of Lemma 3</u>: For every w in B_2 there exists an associated Cauchy sequence $\{w_n\}$, $w_n \in C_2$ such that

$$\begin{array}{l} w_{nt} \text{ is a Cauchy sequence in } C(\Omega) \\ \qquad \text{under the norm } | \ |_{Lm} \end{array} \tag{v}$$

and

$$w_n(x,t) = w_n(x,0) + \int_0^t w_{nt}(x,\nu)d\nu \ . \tag{vi}$$

We know from Lemma 1 and (v) that there exists a $g(x,t)$ such that $\lim_n |w_{nt}-g|_{Lm} = 0$. We now show there exists a subsequence n_j such that

$$\lim_{n_j} \int_0^t w_{n_j t}(x,\nu)d\nu = \int_0^t g(x,\nu)d\nu \tag{vii}$$

for every $t \in [0,T]$ and almost every $x \in [0,1]$.

The inequalities

$$|\int_0^t (w_{nt}(x,\nu)-g(x,\nu)d\nu| \leq \int_0^T |w_{nt}(x,t)-g(x,t)|dt \equiv G_n(x)$$

and

$$|G_n(x)|_L \leq \int_0^T |w_{nt}-g|_L \, dt \leq T|w_{nt}-g|_{Lm} \to 0$$

give

$$\lim_n |G_n(x)|_L = 0 .$$

The Riesz-Fischer theorem asserts the existence of a subsequence n_j such that $G_{n_j}(x) \to 0$ almost everywhere. Therefore (vii) is true. The subsequence $\{w_{n_j}\}$ satisfies (vi), that is

$$w_{n_j}(x,t) = w_{n_j}(x,0) + \int_0^t w_{n_j t}(x,\nu)d\nu , \qquad \text{(viii)}$$

so, if we make use of (vii) and the uniform convergence of $w_{n_j}(x,t)$, the pointwise limit of (viii) yields

$$w(x,t) = w(x,0) + \int_0^t g(x,\nu)d\nu \qquad \text{(ix)}$$

for almost every x in $[0,1]$. Therefore $w(x,t)$ is an indefinite integral in t for almost every x.

The properties of $u \in B$ can now be stated as a

<u>Theorem</u> (<u>Properties of B</u>):

If $u \in B$, then

a. $u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, u_{xxx}$ and u_{xxt} are continuous,

b. u_{tt}, u_{xxx} and u_{xxt} are indefinite integrals in x for every t ,

c. u_{xt} and u_{xxx} are indefinite integrals in t for almost every x .

<u>Proof</u>: Lemma 2 asserts b and Lemma 3 asserts c .

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